# 1 | Some Set Theory

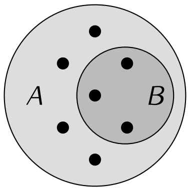
**Sets.**

Frequently used sets:

$\varnothing$ = the empty set (i.e. the set that contains no elements)
$\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ the set of integers
$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ the set of positive integers
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{R}$ = the set of real numbers

We will write $x \in A$ to denote that $x$ is an element of the set $A$ and $y \notin A$ to indicate that $y$ is not an element of $A$.

**1.1 Definition.** A set $B$ is a *subset* of a set $A$ if every element of $B$ is in $A$. In such case we write $B \subseteq A$.




A set $B$ is a *proper subset* of $A$ if $B \subseteq A$ and $B \neq A$.

**1.3 Example.** Here are some often used subsets of $\mathbb{R}$:

1) an open interval:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$


2) a closed interval:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$


3) a half open interval:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

**1.4 Definition.** The *union* of sets $A$ and $B$ is the set $A \cup B$ that consists of all elements that belong to either $A$ or $B$:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The *intersection* of sets $A$ and $B$ is the set $A \cap B$ that consists of all elements that belong to both $A$ and $B$:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

**1.7 Definition.** If $A \cap B = \varnothing$ then we say that $A$ and $B$ are *disjoint sets*.

Definition 1.4 can be extended to unions and intersections of arbitrary families of sets. If $\{A_i\}_{\in I}$ is a family of sets then

$$\bigcup_{i\in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$
$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

**1.10 Definition.** The *difference* of sets $A$ and $B$ is the set $A \smallsetminus B$ consisting of the elements of $A$ that do not belong to $B$:

$$A \smallsetminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

**1.12 Definition.** If $A \subseteq B$ then the set $B \smallsetminus A$ is called the *complement* of $A$ in $B$.

**1.13 Properties of the algebra of sets.**

Distributivity:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

De Morgan's Laws:

$$A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$$
$$A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$$

**1.14 Definition.** The *Cartesian product* of sets $A$, $B$ is the set consisting of all ordered pairs of elements of $A$ and $B$:

$$A \times B = \{(a, b) \mid a \in A, \quad b \in B\}$$

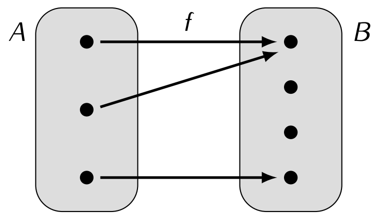
**1.16 Notation.** Given a set $A$ by $A^n$ we will denote the $n$-fold Cartesian product of $A$:

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

### 1.18 Infinite products.

**1.27 Definition.** Let $A$, $B$ be sets

1) A function $f: A \to B$ is *1-1* if $f(x) = f(x')$ only if $x = x'$.



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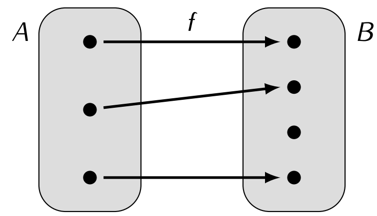

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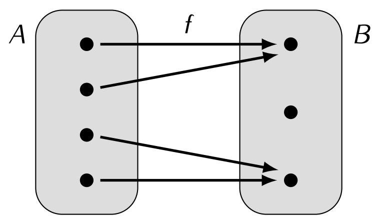
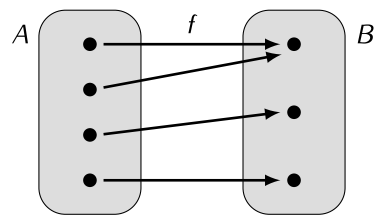
2) A function $f: A \to B$ is *onto* if for every $y \in B$ there is $x \in A$ such that $f(x) = y$

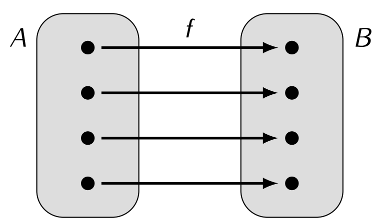


not onto



onto

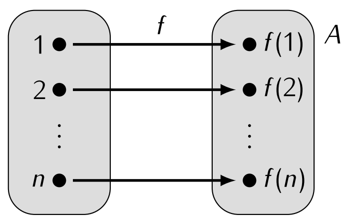
3) A function $f: A \to B$ is a *bijection* if $f$ is both 1-1 and onto.



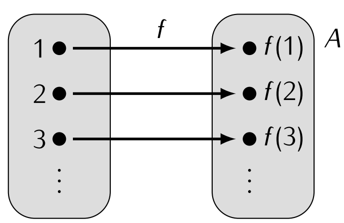
bijection

**1.29 Definition.** Sets $A$, $B$ *have the same cardinality* if there exists a bijection $f\colon A \to B$. In such case we write $|A| = |B|$.

**1.30 Definition.** A set $A$ is *finite* if either $A = \varnothing$ or A has the same cardinality as the set $\{1, \dots, n\}$ for some $n \geq 1$.




**1.31 Definition.** A set $A$ is *infinitely countable* if it is has the same cardinality as the set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$




**1.32 Definition.** A set $A$ is *countable* if it is either finite or infinitely countable.

**1.33 Example.** The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is countable since we have a bijection $f\colon \mathbb{Z}^+ \to \mathbb{N}$ given by $f(k) = k - 1$.

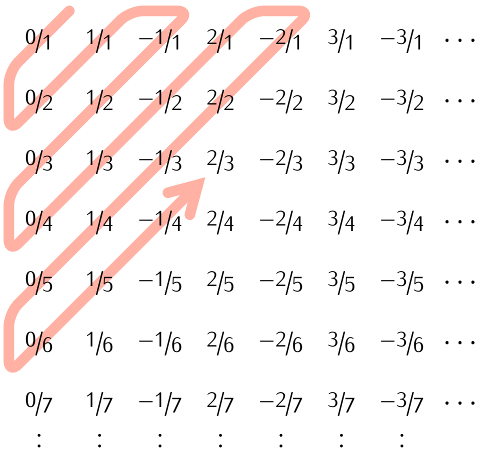
**1.34 Example.** The set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is countable since we have a bijection $f\colon \mathbb{Z}^+ \to \mathbb{Z}$ given by

$$f(k) = \begin{cases} k/2 & \text{if } k \text{ is even} \\ (1-k)/2 & \text{if } k \text{ is odd} \end{cases}$$

In other words:

$$f(1) = 0,\ f(2) = 1,\ f(3) = -1,\ f(4) = 2,\ f(5) = -2,\ f(6) = 3,\ \dots$$

**1.35 Example.** The set of rational numbers $\mathbb{Q}$ is countable. A bijection $f\colon \mathbb{Z}^+ \to \mathbb{Q}$ can be constructed as follows:




$$\begin{aligned}
{}^0\!/_1 &= f(1)\\
{}^0\!/_2 = {}^0\!/_1 &= f(1)\\
{}^1\!/_1 &= f(2)\\
{}^{-1}\!/_1 &= f(3)\\
{}^1\!/_2 &= f(4)\\
{}^0\!/_3 = {}^0\!/_1 &= f(1)\\
{}^0\!/_4 = {}^0\!/_1 &= f(1)\\
{}^1\!/_3 &= f(5)\\
{}^{-1}\!/_2 &= f(6)\\
{}^2\!/_1 &= f(7)\\
\cdots &\quad \cdots
\end{aligned}$$

**1.36 Theorem.** *1) If $A$ is a countable set and $B \subseteq A$ then $B$ is countable.*
*2) If $\{A_1, A_2, \ldots\}$ is a collection of countably many countable sets then the set $\bigcup_{i=1}^{\infty} A_i$ is countable.*
*3) If $\{A_1, A_2, \ldots, A_n\}$ is a collection of finitely many countable sets then the set $A_1 \times \cdots \times A_n$ is countable.*

**1.37 Example.** The set of all real numbers in the interval (0, 1) is not countable.

$$\begin{aligned} f(1) &= 0.31415\dots \\ f(2) &= 0.12345\dots \\ f(3) &= 0.75149\dots \\ f(4) &= 0.00032\dots \\ f(5) &= 0.11111\dots \\ \dots & \quad \dots\ \dots\ \dots \end{aligned}$$

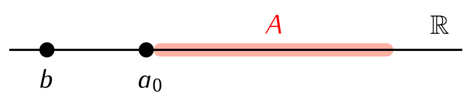
**1.38 Example.** The function $f\colon (0,1) \to \mathbb{R}$ given by $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection. It follows that $|\mathbb{R}| = |(0,1)|$, and so the set $\mathbb{R}$ is not countable.

**Infima and Suprema.**

**1.40 Definition.** Let $A \subseteq \mathbb{R}$. The set $A$ is *bounded below* if there exists a number $b$ such that $b \leq x$ for all $x \in A$. The set $A$ is *bounded above* if there exists a number $c$ such that $x \leq c$ for all $x \in A$. The set $A$ is *bounded* if it is both bounded below and bounded above.

**1.41 Definition.** Let $A \subseteq \mathbb{R}$. If the set $A$ is bounded below then the *greatest lower bound* of $A$ (or *infimum* of $A$) is a number $a_0 \in \mathbb{R}$ such that:

1) $a_0 \leq x$ for all $x \in A$
2) if $b \leq x$ for all $x \in A$ then $b \leq a_0$



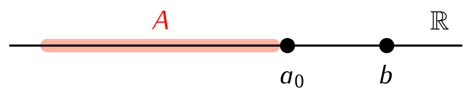

We write: $a_0 = \inf A$.

If the set $A$ is not bounded below then we set $\inf A := -\infty$.

**1.43 Theorem.** *For any non-empty bounded below subset* $A \subseteq \mathbb{R}$ *the number* $\inf A$ *exists.*

**1.44 Definition.** Let $A \subseteq \mathbb{R}$. If the set $A$ is bounded above then the *least upper bound* of $A$ (or *supremum* of $A$) is a number $a_0 \in \mathbb{R}$ such that:

1) $x \leq a_0$ for all $x \in A$
2) if $x \leq b$ for all $x \in A$ then $a_0 \leq b$




We write: $a_0 = \sup A$.

If the set $A$ is not bounded above then we set $\sup A := +\infty$.

**1.46 Theorem.** *For any non-empty bounded above subset $A \subseteq \mathbb{R}$ the number $\sup A$ exists.*