# 12 | Urysohn Metrization Theorem

**12.1 Urysohn Metrization Theorem.** *Every second countable normal space is metrizable.*

**12.2 Definition.** A continuous function $i\colon X \to Y$ is an *embedding* if its restriction $i\colon X \to i(X)$ is a homeomorphism (where $i(X)$ has the topology of a subspace of $Y$).

**12.5 Lemma.** *If $j\colon X \to Y$ is an embedding and $Y$ is a metrizable space then $X$ is also metrizable.*

**12.6 Definition.** Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The *product topology* on $\prod_{i\in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{\prod_{i\in I} U_i \mid U_i \text{ is open in } X_i \text{ and } U_i \neq X_i \text{ for finitely many indices } i \text{ only}\right\}$$

**12.8 Proposition.** *Let* $\{X_i\}_{i\in I}$ *be a family of topological spaces and for* $j \in I$ *let*

$$p_j \colon \prod_{i\in I} X_i \to X_j$$

*be the projection onto the* $j$*-th factor:* $p_j((x_i)_{i\in I}) = x_j$*. Then:*

1) *for any* $j \in I$ *the function* $p_j$ *is continuous.*
2) *A function* $f \colon Y \to \prod_{i\in I} X_i$ *is continuous if and only if the composition* $p_j f \colon Y \to X_j$ *is continuous for all* $j \in I$

*Proof.* Exercise. □

**12.10 Proposition.** *If $X_1, \dots, X_n$ are metrizable spaces then $X_1 \times \cdots \times X_n$ is also a metrizable space.*

**12.11 Proposition.** *If $\{X_i\}_{i=1}^{\infty}$ is an infinite countable family of metrizable spaces then $\prod_{i=1}^{\infty} X_i$ is also a metrizable space.*

**12.12 Example.** The *Hilbert cube* is the topological space $[0, 1]^{\aleph_0}$ obtained as the infinite countable product of the closed interval $[0, 1]$:

$$[0, 1]^{\aleph_0} = \prod_{i=1}^{\infty} [0, 1]$$

Elements of $[0, 1]^{\aleph_0}$ are infinite sequences $(t_i) = (t_1, t_2, \dots)$ where $t_i \in [0, 1]$ for $i = 1, 2, \dots$ The Hilbert cube is a metric space with a metric $\varrho$ given by

$$\varrho((t_i), (s_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} |t_i - s_i|$$

**12.13 Definition.** Let $X$ be a topological space and let $\{f_i\}_{i\in I}$ be a family of continuous functions $f_i\colon X \to [0,1]$. We say that the family $\{f_i\}_{i\in I}$ *separates points from closed sets* if for any point $x_0 \in X$ and any closed set $A \subseteq X$ such that $x_0 \notin A$ there is a function $f_j \in \{f_i\}_{i\in I}$ such that $f_j(x_0) > 0$ and $f_j|_A = 0$.

**12.14 Embedding Lemma.** *Let $X$ be a $T_1$-space. If $\{f_i\colon X \to [0,1]\}_{i\in I}$ is a family that separates points from closed sets then the map*

$$f_\infty\colon X \to \prod_{i\in I} [0,1]$$

*given by $f_\infty(x) = (f_i(x))_{i\in I}$ is an embedding.*

*Proof of Theorem 12.1.*

□

**12.16 Proposition.** *Every second countable regular space is normal.*

*Proof.* Exercise. □

**12.17 Urysohn Metrization Theorem (v.2).** *Every second countable regular space is metrizable.*

**12.18 Definition.** Let $X$ be a topological space. A collection $\mathcal{U} = \{U_i\}_{i\in I}$ of open sets in $X$ is *locally finite* if each point $x \in X$ has an open neighborhood $V_x$ such that $V_x \cap U_i \neq \varnothing$ for finitely many $i \in I$ only.

A collection $\mathcal{U}$ is *countably locally finite* if it can be decomposed into a countable union $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ where each collection $\mathcal{U}_n$ is locally finite.

**12.19 Nagata-Smirnov Metrization Theorem.** *Let $X$ be a topological space. The following conditions are equivalent:*

1) *$X$ is metrizable.*
2) *$X$ is regular and it has a basis which is countably locally finite.*