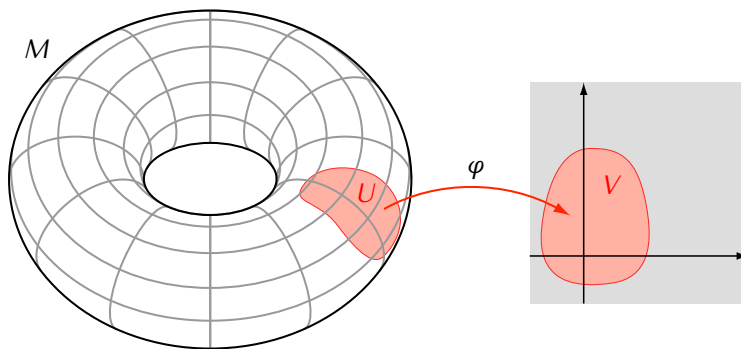


13 | Metrization of Manifolds

13.1 Definition. A *topological manifold of dimension n* is a topological space M which is a Hausdorff, second countable, and such that every point of M has an open neighborhood homeomorphic to an open subset of \mathbb{R}^n (we say that M is *locally homeomorphic to \mathbb{R}^n*).



13.3 Lemma. If M is an n -dimensional manifold then:

- 1) for any point $x \in M$ there exists a coordinate chart $\varphi: U \rightarrow V$ such that $x \in U$, V is an open ball $V = B(y, r)$, and $\varphi(x) = y$;
- 2) for any point $x \in M$ there exists a coordinate chart $\psi: U \rightarrow V$ such that $x \in U$, $V = \mathbb{R}^n$, and $\psi(x) = 0$.

Proof. Exercise. □

13.4 Example. A space M is a manifold of dimension 0 if and only if M is a countable (finite or infinite) discrete space.

13.5 Example. If U is an open set in \mathbb{R}^n then U is an n -dimensional manifold. The identity map $\text{id}: U \rightarrow U$ is then a coordinate chart defined on the whole manifold U . In particular \mathbb{R}^n is an n -dimensional manifold.

13.6 Example. The n -dimensional sphere

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

is an n -dimensional manifold.

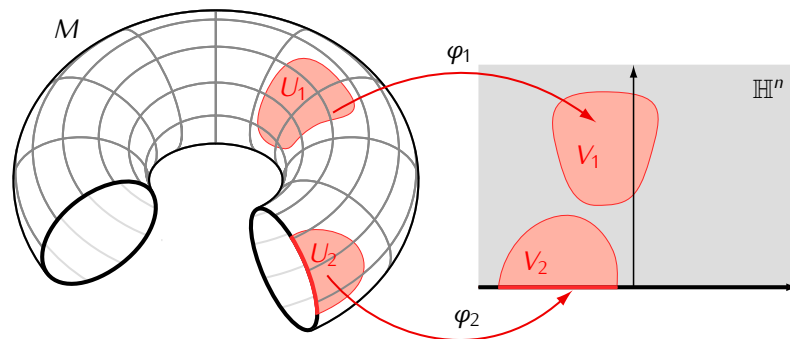
13.7 Proposition. *If M is an m -dimensional manifold and N is an n -dimensional manifold then $M \times N$ is an $m + n$ -dimensional manifold.*

Proof. Exercise. □

13.9 Note. There exist topological spaces that are locally homeomorphic to \mathbb{R}^n , but do not satisfy the other conditions of the definition of a manifold (13.1).

13.10 Invariance of Dimension Theorem. *If M is a non-empty topological space such that M is a manifold of dimension m and M is also a manifold of dimension n then $m = n$.*

13.11 Definition. A *topological n -dimensional manifold with boundary* is a topological space M which is a Hausdorff, second countable, and such that every point of M has an open neighborhood homeomorphic to an open subset of \mathbb{H}^n .



13.13 Theorem. Let M be an n -dimensional manifold with boundary, let $x_0 \in M$ and let $\varphi: U \rightarrow V$ be a local coordinate chart such that $x_0 \in U$. If $\varphi(x_0) \in \partial\mathbb{H}^n$ then for any other local coordinate chart $\psi: U' \rightarrow V'$ such that $x_0 \in U'$ we have $\psi(x_0) \in \partial\mathbb{H}^n$.

13.14 Definition. Let M be a manifold with boundary. The subspace of M consisting of all boundary points of M is called *the boundary of M* and it is denoted by ∂M .

13.15 Example. The space \mathbb{H}^n is trivially an n -dimensional manifold with boundary.

13.16 Example. For any n the closed n -dimensional ball

$$\bar{B}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$$

is an n -dimensional manifold with boundary (exercise). In this case we have $\partial\bar{B}^n = S^{n-1}$.

13.17 Example. If M is a manifold (without boundary) then we can consider it as a manifold with boundary, where $\partial M = \emptyset$.

13.18 Example. If M is an m -dimensional manifold with boundary and N is an n -dimensional manifold without boundary then $M \times N$ is an $(m + n)$ -dimensional manifold with boundary (exercise).

13.20 Theorem. *Every topological manifold (with or without boundary) is metrizable.*

13.21 Lemma. *Let M be an n -dimensional topological manifold, and let $\varphi: U \rightarrow V$ be a coordinate chart on M . If $\bar{B}(x, r)$ is a closed ball in \mathbb{R}^n such that $\bar{B}(x, r) \subseteq V$ then the set $\varphi^{-1}(\bar{B}(x, r))$ is closed in M .*

