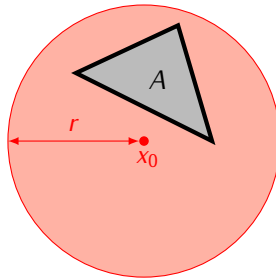


15 | Heine-Borel Theorem

15.1 Definition. Let (X, ρ) be a metric space. A set $A \subseteq X$ is *bounded* if there exists an open ball $B(x_0, r) \subseteq X$ such that $A \subseteq B(x_0, r)$.



15.2 Proposition. Let (X, ρ) be a metric space and let $A \subseteq X$. The following conditions are equivalent:

- 1) A is bounded.
- 2) For each $x \in X$ there exists $r_x > 0$ such that $A \subseteq B(x, r_x)$.
- 3) There exists $R > 0$ such that $\rho(x_1, x_2) < R$ for all $x_1, x_2 \in A$.

Proof. Exercise. □

15.3 Heine-Borel Theorem. A set $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.

15.5 Theorem. *If X, Y are compact spaces then the space $X \times Y$ is also compact.*

15.6 Corollary. *If X_1, \dots, X_n are compact spaces then the space $X_1 \times \dots \times X_n$ is compact.*

15.7 Corollary. *For $i = 1, \dots, n$ let $[a_i, b_i] \subseteq \mathbb{R}$ be a closed interval. The closed box*

$$[a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

is compact.

Proof of Theorem 15.3. (\Rightarrow) Exercise.

(\Leftarrow)

□