

16 | Compact Metric Spaces

16.1 Definition. A topological space X is *sequentially compact* if every sequence $\{x_n\} \subseteq X$ contains a convergent subsequence.

16.2 Theorem. A metric space (X, ρ) is compact if and only if it is sequentially compact.

16.4 Lemma. *Let (X, ρ) be a metric space. If a sequence $\{x_n\} \subseteq X$ does not contain any convergent subsequence then $\{x_n\}$ is a closed set in X .*

Proof. Exercise. □

16.5 Lemma. *Let (X, ρ) be a metric space. If a sequence $\{x_n\} \subseteq X$ does not contain any convergent subsequence then for each $k = 1, 2, \dots$ there exists $\varepsilon_k > 0$ such that $B(x_k, \varepsilon_k) \cap \{x_n\} = x_k$.*

Proof. Exercise. □

Proof of Theorem 16.2 (\Rightarrow).

□

16.6 Definition. Let (X, ρ) be a metric space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X . A *Lebesgue number* for \mathcal{U} is a number $\lambda_{\mathcal{U}} > 0$ such that for every $x \in X$ we have $B(x, \lambda_{\mathcal{U}}) \subseteq U_i$ for some $U_i \in \mathcal{U}$.

16.8 Lemma. *If (X, ρ) is a sequentially compact metric space then for any open cover \mathcal{U} of X there exists a Lebesgue number for \mathcal{U} .*

16.9 Definition. Let (X, ρ) be a metric space. For $\varepsilon > 0$ an ε -net in X is a set of points $\{x_i\}_{i \in I} \subseteq X$ such that $X = \bigcup_{i \in I} B(x_i, \varepsilon)$.

16.11 Lemma. *Let (X, ρ) be a sequentially compact metric space. For every $\varepsilon > 0$ there exists a finite ε -net in X .*

Proof of Theorem 16.2 (\Leftarrow).

□

16.12 Corollary. *If (X, ρ) is a compact metric space then for any open cover \mathcal{U} of X there exists a Lebesgue number for \mathcal{U} .*