

# 18 | Compactification

**18.1 Proposition.** *Let  $X$  be a topological space. If there exists an embedding  $j: X \rightarrow Y$  such that  $Y$  is a compact Hausdorff space then there exists an embedding  $j_1: X \rightarrow Z$  such that  $Z$  is compact Hausdorff and  $\overline{j_1(X)} = Z$ .*

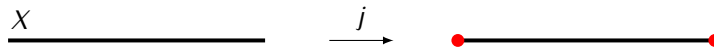
**18.2 Definition.** A space  $Z$  is a *compactification* of  $X$  if  $Z$  is compact Hausdorff and there exists an embedding  $j: X \rightarrow Z$  such that  $\overline{j(X)} = Z$ .

**18.3 Corollary.** *Let  $X$  be a topological space. The following conditions are equivalent:*

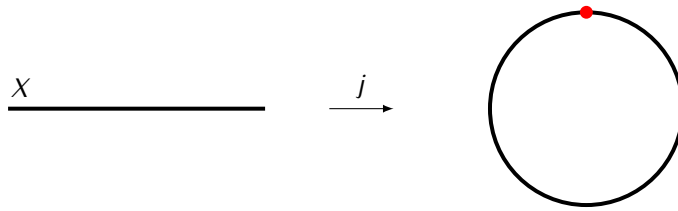
- 1) *There exists a compactification of  $X$ .*
- 2) *There exists an embedding  $j: X \rightarrow Y$  where  $Y$  is a compact Hausdorff space.*

*Proof.* Follows from Proposition 18.1. □

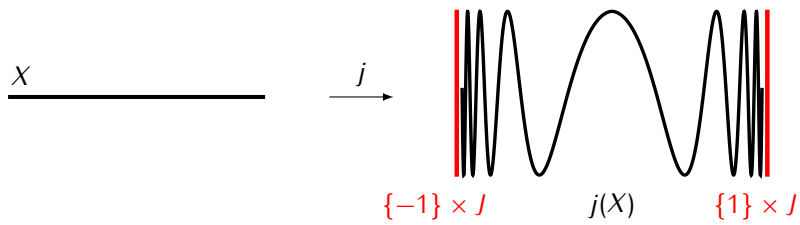
18.4 Example.



18.5 Example.



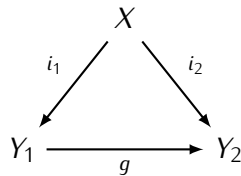
18.6 Example.



**18.7 Theorem.** *A space  $X$  has a compactification if and only if  $X$  is completely regular (i.e. it is a  $T_{3\frac{1}{2}}$ -space).*

**18.9 Definition.** Let  $X$  be a completely regular space and let  $j_X: X \rightarrow \prod_{f \in C(X)} [0, 1]$  be the embedding defined in the proof of Theorem 18.7 and let  $\beta(X)$  be the closure of  $j_X(X)$  in  $\prod_{f \in C(X)} [0, 1]$ . The compactification  $j_X: X \rightarrow \beta(X)$  is called the *Čech-Stone compactification* of  $X$ .

**18.10 Definition.** Let  $X$  be a space and let  $i_1: X \rightarrow Y_1$ ,  $i_2: X \rightarrow Y_2$  be compactifications of  $X$ . We will write  $Y_1 \geq Y_2$  if there exists a continuous function  $g: Y_1 \rightarrow Y_2$  such that  $i_2 = gi_1$ :



**18.11 Proposition.** Let  $i_1: X \rightarrow Y_1$ ,  $i_2: X \rightarrow Y_2$  be compactifications of a space  $X$ .

- 1) If  $Y_1 \geq Y_2$  then there exists only one map  $g: Y_1 \rightarrow Y_2$  satisfying  $i_2 = gi_1$ . Moreover  $g$  is onto.
- 2)  $Y_1 \geq Y_2$  and  $Y_2 \geq Y_1$  if and only if the map  $g: Y_1 \rightarrow Y_2$  is a homeomorphism.

*Proof.* Exercise. □

**18.12 Theorem.** *Let  $X$  be a completely regular space and let  $j_X: X \rightarrow \beta(X)$  be the Čech-Stone compactification of  $X$ . For any compactification  $i: X \rightarrow Y$  of  $X$  we have  $\beta(X) \geq Y$ .*

**18.13 Lemma.** *If  $f: X_1 \rightarrow X_2$  is a continuous map of compact Hausdorff spaces then  $f(\overline{A}) = \overline{f(A)}$  for any  $A \subseteq X_1$ .*

*Proof.* Exercise. □

**18.14 Definition.** A space  $Z$  is a *one-point compactification* of a space  $X$  if  $Z$  is a compactification of  $X$  with embedding  $j: X \rightarrow Z$  such that the set  $Z \setminus j(X)$  consists of only one point.

**18.16 Proposition.** *If a space  $X$  has a one-point compactification  $j: X \rightarrow Z$  then this compactification is unique up to homeomorphism. That is, if  $j': X \rightarrow Z'$  is another one-point compactification of  $X$  then there exists a homeomorphism  $h: Z \rightarrow Z'$  such that  $j' = hj$ .*

*Proof.* Exercise. □

**18.17 Definition.** A topological space  $X$  is *locally compact* if every point  $x \in X$  has an open neighborhood  $U_x \subseteq X$  such that the closure  $\overline{U_x}$  is compact.

**18.19 Theorem.** *Let  $X$  be a non-compact topological space. The following conditions are equivalent:*

- 1) The space  $X$  is locally compact and Hausdorff.*
- 2) There exists a one-point compactification of  $X$ .*

**18.20 Corollary.** *If  $X$  is a locally compact Hausdorff space then  $X$  is completely regular.*

*Proof.* Follows from Theorem 18.7 and Theorem 18.19.

□

**18.21 Corollary.** *Let  $X$  be a topological space. The following conditions are equivalent:*

- 1) *The space  $X$  is locally compact and Hausdorff .*
- 2) *There exists an embedding  $i: X \rightarrow Y$  where  $Y$  is compact Hausdorff space and  $i(X)$  is an open set in  $Y$ .*

**18.22 Proposition.** *Let  $X$  be a non-compact, locally compact Hausdorff space and let  $j: X \rightarrow X^+$  be the one-point compactification of  $X$ . For every compactification  $i: X \rightarrow Y$  of  $X$  we have  $Y \geq X^+$ .*

*Proof.* Exercise.

□