

19 | Quotient Spaces

19.1 Definition. Let X be a set. An *equivalence relation on X* is a binary relation \sim satisfying three properties:

- 1) $x \sim x$ for all $x \in X$ (reflexivity)
- 2) if $x \sim y$ then $y \sim x$ (symmetry)
- 3) if $x \sim y$ and $y \sim z$ then $x \sim z$ (transitivity)

19.4 Definition. Let X be a set with an equivalence relation \sim and let $x \in X$. The *equivalence class* of x is the subset $[x] \subseteq X$ consisting of all elements that are in the relation with x :

$$[x] = \{y \in X \mid x \sim y\}$$

19.7 Proposition. Let X be a set with an equivalence relation \sim , and let $x, y \in X$.

- 1) If $x \sim y$ then $[x] = [y]$.
- 2) If $x \not\sim y$ then $[x] \cap [y] = \emptyset$.

19.9 Definition. Let X be a set with an equivalence relation \sim . The *quotient set* of X is the set X/\sim whose elements are all distinct equivalence classes of \sim . The function

$$\pi: X \rightarrow X/\sim$$

given by $\pi(x) = [x]$ is called the *quotient map*.

19.11 Definition. Let X be a topological space and let \sim be an equivalence relation on X . The *quotient topology* on the set X/\sim is the topology where a set $U \subseteq X/\sim$ is open if the set $\pi^{-1}(U)$ is open in X . The set X/\sim with this topology is called the *quotient space* of X taken with respect to the relation \sim .

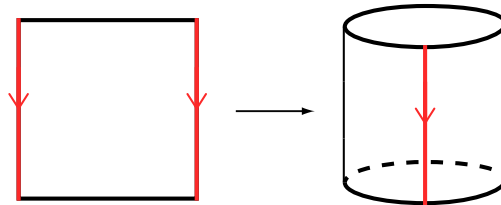
19.12 Proposition. Let X be a topological space and let \sim be an equivalence relation on X . A set $A \subseteq X/\sim$ is closed if and only if the set $\pi^{-1}(A)$ is closed in X .

Proof. Exercise. □

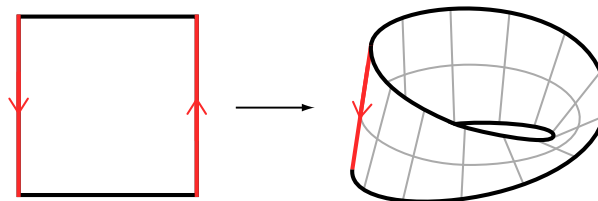
19.13 Proposition. Let X, Y be topological spaces and let \sim be an equivalence relation on X . A function $f: X/\sim \rightarrow Y$ is continuous if and only if the function $f\pi: X \rightarrow Y$ is continuous.

Proof. Exercise. □

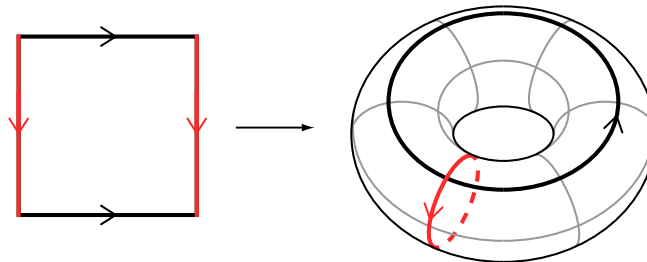
19.17 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation defined as in Example 19.2: $(0, t) \sim (1, t)$ for all $t \in [0, 1]$. Using arguments similar as in Example 19.15 we can show that the quotient space is homeomorphic to the cylinder $S^1 \times [0, 1]$:



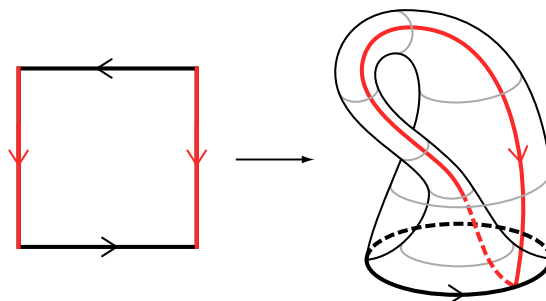
19.18 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, 1 - t)$ for all $t \in [0, 1]$. The space obtained as a quotient space is called the *Möbius band*:



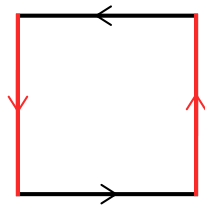
19.19 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, t)$ for all $t \in [0, 1]$ and $(s, 0) \sim (s, 1)$ for all $s \in [0, 1]$. Using arguments similar to these given in Example 19.15 one can show that the quotient space in this case is homeomorphic to the torus:



19.20 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, t)$ for all $t \in [0, 1]$ and $(s, 0) \sim (1 - s, 1)$ for all $s \in [0, 1]$. The resulting quotient space is called the *Klein bottle*. One can show that the Klein bottle is a two dimensional manifold.



19.21 Example. Following the scheme of the last two examples we can consider the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, 1 - t)$ and $(s, 0) \sim (1 - s, 1)$ for all $s, t \in [0, 1]$:



Disjoint unions

