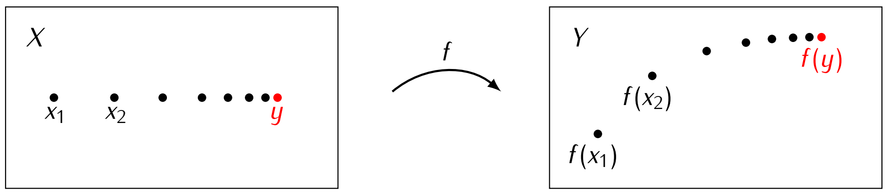
6 | Continuous Functions

6.1 Proposition. *Let X, Y be topological spaces. A function $f\colon X \to Y$ is continuous if and only if for every closed set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is closed.*

6.2 Proposition. *Let (X, ϱ) be a metric space, let Y be a topological space, and let $f\colon X \to Y$ be a function. The following conditions are equivalent:*

1) *f is continuous.*
2) *For any sequence $\{x_n\} \subseteq X$ if $x_n \to y$ for some $y \in X$ then $f(x_n) \to f(y)$.*



6.3 Proposition. *Let $f \colon X \to Y$ be a continuous function of topological spaces. If $\{x_n\} \subseteq X$ is a sequence and $x_n \to x$ for some $x \in X$ then $f(x_n) \to f(x)$.*

Proof. Exercise. □

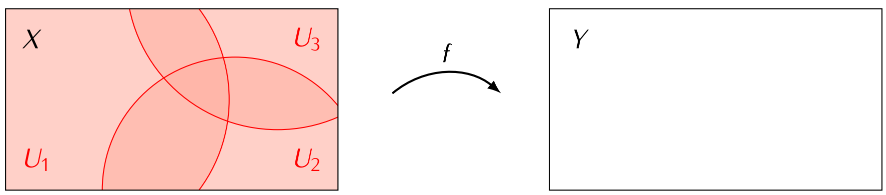
6.4 Example. We will show that the implication 2) ⇒ 1) in Proposition 6.2 is not true if X is a general topological space. Let X be the space defined in Example 5.16: $X = \mathbb{R}$ with the topology

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

6.5 Proposition. *If $f: X \to Y$ and $g: Y \to Z$ are continuous functions then the function $gf: X \to Z$ is also continuous.*

Proof. Exercise. $\square$

6.6 Open Pasting Lemma. *Let X, Y be topological spaces and let $\{U_i\}_{i \in I}$ be a family of open sets in X such that $\bigcup_{i \in I} U_i = X$. Assume that for $i \in I$ we have a continuous function $f_i: U_i \to Y$ such that $f_i(x) = f_j(x)$ if $x \in U_i \cap U_j$. Then the function $f: X \to Y$ given by $f(x) = f_i(x)$ for $x \in U_i$ is continuous.*



6.7 Closed Pasting Lemma. *Let X, Y be topological spaces and let $A_1, \dots, A_n \subseteq X$ be a finite family of closed sets such that $\bigcup_{i=1}^n A_i = X$. Assume that for $i = 1, 2, \dots, n$ we have a continuous function $f_i \colon A_i \to Y$ such that $f_i(x) = f_j(x)$ if $x \in A_i \cap A_j$. Then the function $f \colon X \to Y$ given by $f(x) = f_i(x)$ for $x \in A_i$ is continuous.*

Proof. Exercise. □

6.9 Definition. A *homeomorphism* is a continuous function $f\colon X \to Y$ such that f is a bijection and the inverse function $f^{-1}\colon Y \to X$ is continuous.

6.10 Proposition. *1) For any topological space the identify function* $\mathrm{id}_X\colon X \to X$ *given by* $\mathrm{id}_X(x) = x$ *is a homeomorphism.*

2) If $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are homeomorphisms then the function* $gf\colon X \to Z$ *is also a homeomorphism.*

3) If $f\colon X \to Y$ *is a homeomorphism then the inverse function* $f^{-1}\colon Y \to X$ *is also a homeomorphism.*

4) If $f\colon X \to Y$ *is a homeomorphism and* $Z \subseteq X$ *then the function* $f|_Z\colon Z \to f(Z)$ *is also a homeomorphism.*

Proof. Exercise. □

6.12 Proposition. *Let $f\colon X \to Y$ be a continuous bijection. The following conditions are equivalent:*

(i) The function f is a homeomorphism.

(ii) For each open set $U \subseteq X$ the set $f(U) \subseteq Y$ is open.

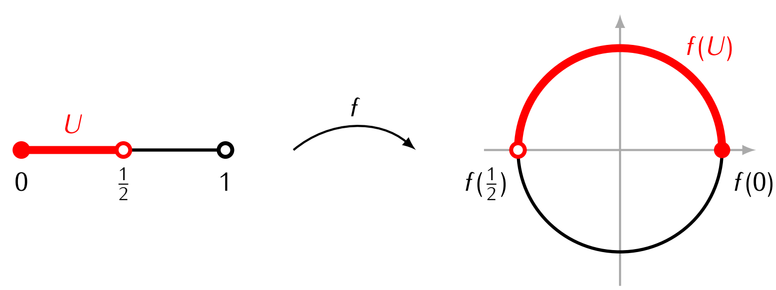
(iii) For each closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed.

Proof. Exercise. $\square$

6.13 Example. Recall that S^1 denotes the unit circle:

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

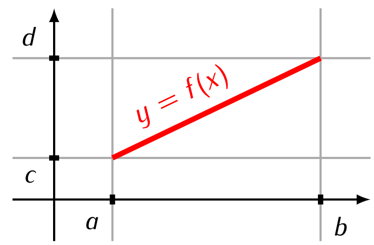
The function $f\colon [0,1) \to S^1$ given by $f(x) = (\cos 2\pi x, \sin 2\pi x)$ is a continuous bijection, but it is not a homeomorphism since the set $U = [0, \frac{1}{2})$ is open in $[0,1)$, but $f(U)$ is not open in S^1.



6.14 Definition. We say that topological spaces X, Y are *homeomorphic* if there exists a homeomorphism $f\colon X \to Y$. In such case we write: $X \cong Y$.

6.16 Example. For any $a < b$ and $c < d$ the open intervals $(a, b), (c, d) \subseteq \mathbb{R}$ are homeomorphic. To see this take e.g. the function $f\colon (a, b) \to (c, d)$ defined by

$$f(x) = \left(\frac{c-d}{a-b}\right)x + \left(\frac{ad-bc}{a-b}\right)$$



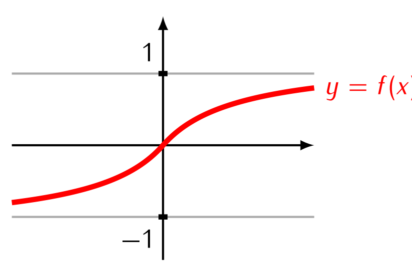
This function is a continuous bijection. Its inverse function $f^{-1}\colon (c, d) \to (a, b)$ is given by

$$f^{-1}(x) = \left(\frac{a-b}{c-d}\right)x + \left(\frac{cb-da}{c-d}\right)$$

so it is also continuous. By the same argument for any $a < b$ and $c < d$ the closed intervals $[a, b], [c, d] \subseteq \mathbb{R}$ are homeomorphic.

6.18 Example. We will show that for any $a < b$ the open interval (a, b) is homeomorphic to $\mathbb{R}$. Since $(a, b) \cong (-1, 1)$ it will be enough to check that $\mathbb{R} \cong (-1, 1)$. Take the function $f\colon \mathbb{R} \to (-1, 1)$ given by

$$f(x) = \frac{x}{1 + |x|}$$



This function is a continuous bijection with the inverse function $f^{-1}\colon (-1, 1) \to \mathbb{R}$ is given by

$$f^{-1}(x) = \frac{x}{1 - |x|}$$

Since f^{-1} is continuous we obtain that f is a homeomorphism.

6.20 Example. We will show that for any point $x_0 \in S^1$ there is a homeomorphism $S^1 \smallsetminus \{x_0\} \cong \mathbb{R}$.