

7 | Connectedness

7.2 Definition. A topological space X is *connected* if for any two open sets $U, V \subseteq X$ such that $U \cup V = X$ and $U, V \neq \emptyset$ we have $U \cap V \neq \emptyset$.

7.3 Definition. If X is a topological space and $U, V \subseteq X$ are non-empty open sets such that $U \cap V = \emptyset$ and $U \cup V = X$ then we say that $\{U, V\}$ is a *separation* of X .

7.5 Proposition. *Let $a < b$. The intervals (a, b) , $[a, b]$, $(a, b]$, and $[a, b)$ are connected topological spaces.*

7.6 Proposition. *If X is a connected subspace of \mathbb{R} then X is an interval (either open, closed, or half-closed, finite or infinite).*

Proof. Exercise.

□

7.8 Proposition. *Let $f: X \rightarrow Y$ be a continuous function. If f is onto and the space X is connected then Y is also connected.*

7.9 Corollary. *If $f: X \rightarrow Y$ is a continuous function and X is a connected space then $f(X)$ is connected.*

7.10 Intermediate Value Theorem. *Let X be a connected topological space and let $f: X \rightarrow \mathbb{R}$ be a continuous function. If $a < b$ are points in \mathbb{R} such that $a = f(x)$ and $b = f(y)$ for some $x, y \in X$ then for each $c \in [a, b]$ there exists $z \in X$ such that $c = f(z)$.*

7.11 Corollary. *If $X \cong Y$ and X is a connected space then Y is also connected.*

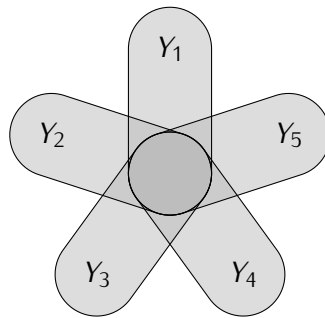
7.13 Note. A *topological invariant* is a property of topological spaces such that if a space X has this property and $X \cong Y$ then Y also has this property. By Corollary 7.11 connectedness is a topological invariant.

7.14 Proposition. Let X be a topological space. The following conditions are equivalent :

- 1) X is connected
- 2) For any closed sets $A, B \subseteq X$ such that $A, B \neq X$ and $A \cap B = \emptyset$ we have $A \cup B \neq X$.
- 3) If $A \subseteq X$ is a set that is both open and closed then either $A = X$ or $A = \emptyset$.
- 4) If $D = \{0, 1\}$ is a space with the discrete topology then any continuous function $f: X \rightarrow D$ is a constant function.

Proof. Exercise. □

7.15 Proposition. Let X be a topological space and for $i \in I$ let Y_i be a subspace of X . Assume that $\bigcup_{i \in I} Y_i = X$ and $\bigcap_{i \in I} Y_i \neq \emptyset$. If Y_i is connected for each $i \in I$ then X is also connected.



7.16 Corollary. The space \mathbb{R}^n is connected for all $n \geq 1$.

7.17 Definition. Let X be a topological space. A *connected component* of X is a subspace $Y \subseteq X$ such that

- 1) Y is connected
- 2) if $Y \subseteq Z \subseteq X$ and Z is connected then $Y = Z$.

7.18 Proposition. Let X be a topological space.

- 1) For every point $x_0 \in X$ there exist a connected component $Y \subseteq X$ such that $x_0 \in Y$.
- 2) If Y, Y' are connected components of X then either $Y \cap Y' = \emptyset$ or $Y = Y'$.

7.19 Corollary. Let X be a topological space. If $Z \subseteq X$ is a connected subspace then there exists a connected component $Y \subseteq X$ such that $Z \subseteq Y$.

Proof. Exercise. □

7.20 Corollary. Let $f: X \rightarrow Y$ be a continuous function. If X is a connected space then there exists a connected component $Z \subseteq Y$ such that $f(X) \subseteq Z$.

Proof. Exercise. □