1 | Some Set Theory

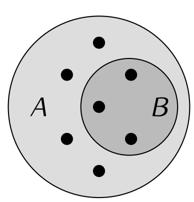
A topological space is a set equipped with some additional structure which, roughly speaking, specifies which elements of the set are close to each other. This lets us define what it means that a function between topological spaces is continuous: intuitively, such function maps elements which are close in one space to elements which are close in the other space. Before we start discussing topological spaces and continuous functions in detail it will be worth go over the basics notions related to sets and functions between sets. This chapter is intended as a quick review of this material. We will also fix here some notation and terminology.

Sets. In general sets will be denoted by capital letters: $A, B, C, \dots$ We will also use the following notation for sets that will be of a particularly interest:

$\varnothing =$ the empty set (i.e. the set that contains no elements)
$\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ the set of integers
$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ the set of positive integers
$\mathbb{Q} =$ the set of rational numbers
$\mathbb{R} =$ the set of real numbers

We will write $x \in A$ to denote that x is an element of the set A and $y \notin A$ to indicate that y is not an element of A. For example, $5 \in \mathbb{Z}$, $\frac{1}{3} \notin \mathbb{Z}$.

1.1 Definition. A set B is a *subset* of a set A if every element of B is in A. In such case we write $B \subseteq A$.



A set B is a *proper subset* of A if $B \subseteq A$ and $B \neq A$.

1.2 Example. $\varnothing \subseteq \mathbb{Z}^+ \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

1.3 Example. Here are some often used subsets of $\mathbb{R}$:

1) an open interval:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

2) a closed interval:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

3) a half open interval:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

1.4 Definition. The *union* of sets A and B is the set $A \cup B$ that consists of all elements that belong to either A or B:

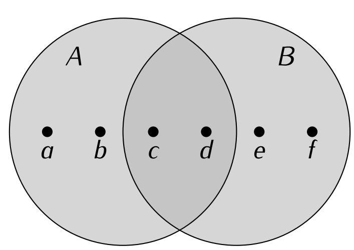
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The *intersection* of sets A and B is the set $A \cap B$ that consists of all elements that belong to both A and B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

1.5 Example. If $A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$

then $A \cup B = \{a, b, c, d, e, f\}$ and $A \cap B = \{c, d\}$.



1.6 Example. If $A = \{a, b, c, d\}$ and $C = \{e, f, g\}$ then $A \cap C = \varnothing$.

1.7 Definition. If $A \cap B = \varnothing$ then we say that A and B are *disjoint sets*.

Definition 1.4 can be extended to unions and intersections of arbitrary families of sets. If $\{A_i\}_{\in I}$ is a family of sets then

$$\bigcup_{i\in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$
$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

1.8 Example. For $n \in \mathbb{Z}$ let $A_n = [n, n+1]$. Then

$$\bigcup_{n\in\mathbb{Z}} A_n = \ldots \cup \ [-2,-1] \ \cup \ [-1,0] \ \cup \ [0,1] \ \cup \ [1,2] \ \cup \ldots = \mathbb{R}$$

1.9 Example. For $n = 1, 2, 3, \ldots$ let $B_n = (-\frac{1}{n}, \frac{1}{n})$. Then

$$\bigcap_{n} B_n = (-1,1) \ \cap \ (-\tfrac{1}{2}, \tfrac{1}{2}) \ \cap \ (-\tfrac{1}{3}, \tfrac{1}{3}) \ \cap \ldots = \{0\}$$

1.10 Definition. The *difference* of sets A and B is the set $A \smallsetminus B$ consisting of the elements of A that do not belong to B:

$$A \smallsetminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

1.11 Example. $A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$

$$A \smallsetminus B = \{a, b\}$$
$$B \smallsetminus A = \{e, f\}$$

1.12 Definition. If $A \subseteq B$ then the set $B \smallsetminus A$ is called the *complement* of A in B.

1.13 Properties of the algebra of sets. Here are some basic formulas involving the operations of sets defined above. We will use them very often.

Distributivity:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

De Morgan's Laws:

$$A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$$
$$A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$$

1.14 Definition. The *Cartesian product* of sets A, B is the set consisting of all ordered pairs of elements of A and B:

$$A \times B = \{(a, b) \mid a \in A, \quad b \in B\}$$

1.15 Example. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$

$$A \times B = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$$

1.16 Notation. Given a set A by A^n we will denote the n-fold Cartesian product of A:

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

1.17 Example.

$$\begin{aligned}\mathbb{R}^2 =& \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \\ \mathbb{R}^3 =& \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}\end{aligned}$$

1.18 Infinite products. Let $A_1, ..., A_n$ be a collection of n sets. Notice that elements of the product $A_1 \times \cdots \times A_n$ can be identified with functions $f\colon \{1, 2, \dots, n\} \to \bigcup_{i=1}^n A_i$ such that $f(i) \in A_i$. Indeed, every such function defines an element $(f(1), f(2), \dots, f(n)) \in A_1 \times \cdots \times A_n$. Conversely, every element $(a_1, \dots, a_n) \in A_1 \times \cdots \times A_n$ defines a function $f\colon \{1, 2, \dots, n\} \to \bigcup_{i=1}^n A_i$ given by $f(i) = a_i$. We can use this observation to define products of an arbitrary (finite or infinite) families of sets. If $\{A_i\}_{i \in I}$ is a family of sets then $\prod_{i \in I} A_i$ is the set consisting of all functions $f\colon I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$.

1.19 Example. for $r \in \mathbb{R}$ let $A_r = [r, r + 1]$. Then $\prod_{r \in \mathbb{R}} A_r$ is the set consisting all functions $f\colon \mathbb{R} \to \bigcup_{r \in \mathbb{R}}[r, r + 1] = \mathbb{R}$ such that $f(r) \in [r, r + 1]$ for all $r \in \mathbb{R}$.

1.20 Note. We will usually denote elements of $\prod_{i \in I} A_i$ by $(a_i)_{i \in I}$. This notation indicates the element defined by the function $f\colon I \to \bigcup_{i \in I} A_i$ given by $f(i) = a_i$.

In many cases given a set A we are interested in describing a relation satisfied by some pairs of elements of the set. Here are some examples of such relations:

1.21 Example. In the set $\mathbb{R}$ of real numbers we can consider the relation "$<$". Numbers $a, b \in \mathbb{R}$ satisfy this relation if $b - a$ is a positive number. We write then $a < b$.

1.22 Example. In the set $\mathbb{Z}$ of integers we can consider the divisibility relation "$|$". Integers $a, b \in \mathbb{Z}$ satisfy this relation if $b = an$ for some $n \in \mathbb{Z}$. In such case we write $a|b$.

1.23 Example. In any set A we can define the equality relation "$=$" which is satisfied by elements $a, b \in A$ only if a and b are the same element.

Formally we define binary relations as follows:

1.24 Definition. A *binary relation* on a set A is a subset $R \subseteq A \times A$. If $(a, b) \in R$ then we write aRb.

1.25 Example. The divisibility relation on the set of integers is the subset $R \subseteq \mathbb{Z} \times \mathbb{Z}$ given by

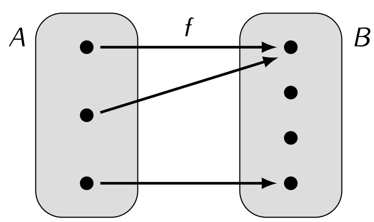
$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b = an \text{ for some } n \in \mathbb{Z}\}$$

1.26 Example. The equality relation on a set A is the subset of $R \subseteq A \times A$ where

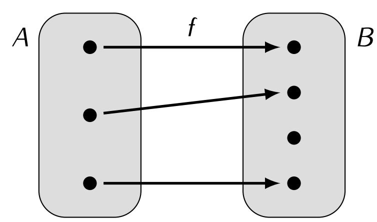
$$R = \{(a, a) \in A \times A \mid a \in A\}$$

1.27 Definition. Let A, B be sets

1) A function $f\colon A \to B$ is *1-1* if $f(x) = f(x')$ only if $x = x'$.



not 1-1



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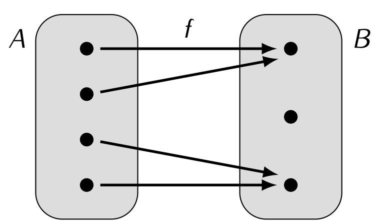
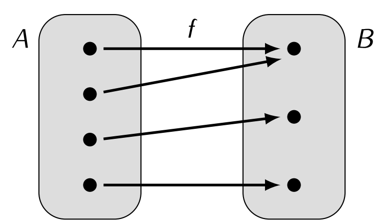
2) A function $f\colon A \to B$ is *onto* if for every $y \in B$ there is $x \in A$ such that $f(x) = y$

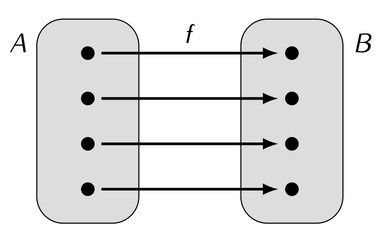


not onto



onto

3) A function $f\colon A \to B$ is a *bijection* if f is both 1-1 and onto.

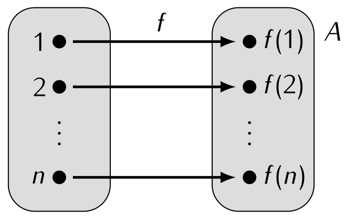


bijection

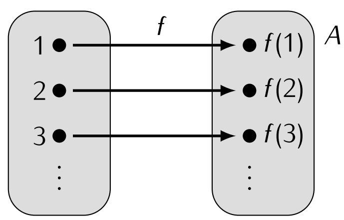
1.28 Note. 1) If $f\colon A \to B$ is a bijection then the inverse function $f^{-1}\colon B \to A$ exists and it is also a bijection.
2) If $f\colon A \to B$ and $g\colon B \to C$ are bijections then the function $gf\colon A \to C$ is also a bijection.

1.29 Definition. Sets A, B *have the same cardinality* if there exists a bijection $f\colon A \to B$. In such case we write $|A| = |B|$.

1.30 Definition. A set A is *finite* if either $A = \varnothing$ or A has the same cardinality as the set $\{1, \dots, n\}$ for some $n \geq 1$.



1.31 Definition. A set A is *infinitely countable* if it is has the same cardinality as the set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$



1.32 Definition. A set A is *countable* if it is either finite or infinitely countable.

1.33 Example. The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is countable since we have a bijection $f\colon \mathbb{Z}^+ \to \mathbb{N}$ given by $f(k) = k - 1$.

1.34 Example. The set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is countable since we have a bijection $f\colon \mathbb{Z}^+ \to \mathbb{Z}$ given by

$$f(k) = \begin{cases} k/2 & \text{if } k \text{ is even} \\ (1-k)/2 & \text{if } k \text{ is odd} \end{cases}$$

In other words:

$$f(1) = 0,\ f(2) = 1,\ f(3) = -1,\ f(4) = 2,\ f(5) = -2,\ f(6) = 3,\ \dots$$

1.35 Example. The set of rational numbers $\mathbb{Q}$ is countable. A bijection $f\colon \mathbb{Z}^+ \to \mathbb{Q}$ can be constructed as follows:

$$
\begin{array}{cccccccc}
0/1 & 1/1 & -1/1 & 2/1 & -2/1 & 3/1 & -3/1 & \cdots \\
0/2 & 1/2 & -1/2 & 2/2 & -2/2 & 3/2 & -3/2 & \cdots \\
0/3 & 1/3 & -1/3 & 2/3 & -2/3 & 3/3 & -3/3 & \cdots \\
0/4 & 1/4 & -1/4 & 2/4 & -2/4 & 3/4 & -3/4 & \cdots \\
0/5 & 1/5 & -1/5 & 2/5 & -2/5 & 3/5 & -3/5 & \cdots \\
0/6 & 1/6 & -1/6 & 2/6 & -2/6 & 3/6 & -3/6 & \cdots \\
0/7 & 1/7 & -1/7 & 2/7 & -2/7 & 3/7 & -3/7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
\qquad
\begin{array}{rcl}
0/1 & = & f(1) \\
0/2 = 0/1 & = & f(1) \\
1/1 & = & f(2) \\
-1/1 & = & f(3) \\
1/2 & = & f(4) \\
0/3 = 0/1 & = & f(1) \\
0/4 = 0/1 & = & f(1) \\
1/3 & = & f(5) \\
-1/2 & = & f(6) \\
2/1 & = & f(7) \\
\ldots & & \ldots
\end{array}
$$

Here are some properties of countable sets:

1.36 Theorem. *1) If A is a countable set and $B \subseteq A$ then B is countable.*
2) If $\{A_1, A_2, \ldots\}$ is a collection of countably many countable sets then the set $\bigcup_{i=1}^{\infty} A_i$ is countable.
3) If $\{A_1, A_2, \ldots, A_n\}$ is a collection of finitely many countable sets then the set $A_1 \times \cdots \times A_n$ is countable.

1.37 Example. The set of all real numbers in the interval (0, 1) is not countable. Indeed, assume by contradiction that there exists a bijection $f\colon \mathbb{Z}^+ \to (0, 1)$. Then we would have:

$$
\begin{aligned}
f(1) &= 0.d_1^1 d_2^1 d_3^1 \ldots \\
f(2) &= 0.d_1^2 d_2^2 d_3^2 \ldots \\
f(3) &= 0.d_1^3 d_2^3 d_3^3 \ldots \\
\ldots \quad & \ldots \quad \ldots
\end{aligned}
$$

where $d_1^k, d_2^k, d_3^k, \ldots$ are digits in the decimal expansion of the number $f(k) \in (0, 1)$. Let $x \in (0, 1)$ be the number defined as follows:

$$x = 0.x_1 x_2 x_3 \ldots$$

where

$$x_i = \begin{cases} 1 & \text{if } d_i^i \neq 1 \\ 2 & \text{if } d_i^i = 1 \end{cases}$$

For example, if we have

$$\begin{aligned} f(1) &= 0.31415\dots \\ f(2) &= 0.12345\dots \\ f(3) &= 0.75149\dots \\ f(4) &= 0.00032\dots \\ f(5) &= 0.11111\dots \\ \dots &\quad \dots\ \dots\ \dots \end{aligned}$$

then

$$x = 0.11212\dots$$

Notice that:

$$\begin{aligned} x \neq f(1) &\quad \text{since } x_1 \neq d^1_1 \\ x \neq f(2) &\quad \text{since } x_2 \neq d^2_2 \\ x \neq f(3) &\quad \text{since } x_3 \neq d^3_3 \\ \dots &\quad \quad \dots \quad \dots \end{aligned}$$

In general $x \neq f(k)$ for all $k \in \mathbb{Z}^+$, and so f is not onto.

1.38 Example. The function $f\colon (0,1) \to \mathbb{R}$ given by $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection. It follows that $|\mathbb{R}| = |(0,1)|$, and so the set $\mathbb{R}$ is not countable.

1.39 Notation. 1) If A is a finite set of n elements then we write $|A| = n$.

2) If $|A| = |\mathbb{Z}^+|$ (i.e. A is an infinitely countable set) then we say that A has the *cardinality aleph naught* and we write $|A| = \aleph_0$.

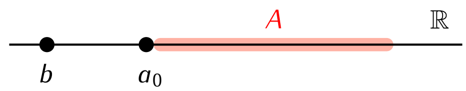
3) If $|A| = |\mathbb{R}|$ then we say that A has the *cardinality of the continuum* and we write $|A| = \mathfrak{c}$.

Infima and Suprema. In the following chapters we will often work with the set $\mathbb{R}$ of real numbers. In particular, we will often use suprema and infima of subsets of $\mathbb{R}$. We conclude this chapter with a quick review of these notions.

1.40 Definition. Let $A \subseteq \mathbb{R}$. The set A is *bounded below* if there exists a number b such that $b \leq x$ for all $x \in A$. The set A is *bounded above* if there exists a number c such that $x \leq c$ for all $x \in A$. The set A is *bounded* if it is both bounded below and bounded above.

1.41 Definition. Let $A \subseteq \mathbb{R}$. If the set A is bounded below then the *greatest lower bound* of A (or *infimum* of A) is a number $a_0 \in \mathbb{R}$ such that:

1) $a_0 \leq x$ for all $x \in A$
2) if $b \leq x$ for all $x \in A$ then $b \leq a_0$



We write: $a_0 = \inf A$.

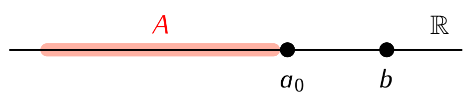
If the set A is not bounded below then we set $\inf A := -\infty$.

1.42 Example.
1) If $A = [0, 1]$ then $\inf A = 0$.
2) If $B = (0, 1)$ then $\inf B = 0$.
3) $\inf \mathbb{Z} = -\infty$

1.43 Theorem. *For any non-empty bounded below subset $A \subseteq \mathbb{R}$ the number $\inf A$ exists.*

1.44 Definition. Let $A \subseteq \mathbb{R}$. If the set A is bounded above then the *least upper bound* of A (or *supremum* of A) is a number $a_0 \in \mathbb{R}$ such that:

1) $x \leq a_0$ for all $x \in A$
2) if $x \leq b$ for all $x \in A$ then $a_0 \leq b$



We write: $a_0 = \sup A$.

If the set A is not bounded above then we set $\sup A := +\infty$.

1.45 Example.
1) If $A = [0, 1]$ then $\sup A = 1$.
2) If $B = (0, 1)$ then $\sup B = 1$.
3) $\sup \mathbb{Z} = +\infty$

1.46 Theorem. *For any non-empty bounded above subset $A \subseteq \mathbb{R}$ the number $\sup A$ exists.*