

# 12 | Urysohn Metrization Theorem

In this chapter we return to the problem of determining which topological spaces are metrizable i.e. can be equipped with a metric which is compatible with their topology. We have seen already that any metrizable space must be normal, but that not every normal space is metrizable. We will show, however, that if a normal space satisfies one extra condition then it is metrizable. Recall that a space  $X$  is second countable if it has a countable basis. We have:

**12.1 Urysohn Metrization Theorem.** *Every second countable normal space is metrizable.*

The main idea of the proof is to show that any space as in the theorem can be identified with a subspace of some metric space. To make this more precise we need the following:

**12.2 Definition.** A continuous function  $i: X \rightarrow Y$  is an *embedding* if its restriction  $i: X \rightarrow i(X)$  is a homeomorphism (where  $i(X)$  has the topology of a subspace of  $Y$ ).

**12.3 Example.** The function  $i: (0, 1) \rightarrow \mathbb{R}$  given by  $i(x) = x$  is an embedding. The function  $j: (0, 1) \rightarrow \mathbb{R}$  given by  $j(x) = 2x$  is another embedding of the interval  $(0, 1)$  into  $\mathbb{R}$ .

**12.4 Note.** 1) If  $j: X \rightarrow Y$  is an embedding then  $j$  must be 1-1.

2) Not every continuous 1-1 function is an embedding. For example, take  $\mathbb{N} = \{0, 1, 2, \dots\}$  with the discrete topology, and let  $f: \mathbb{N} \rightarrow \mathbb{R}$  be given

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} & \text{if } n > 0 \end{cases}$$

The function  $f$  is continuous and it is 1-1, but it is not an embedding since  $f: \mathbb{N} \rightarrow f(\mathbb{N})$  is not a homeomorphism.

**12.5 Lemma.** *If  $j: X \rightarrow Y$  is an embedding and  $Y$  is a metrizable space then  $X$  is also metrizable.*

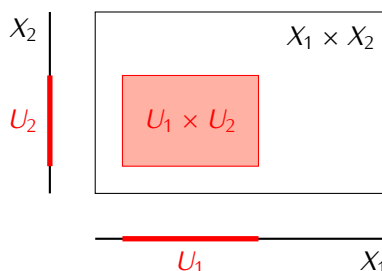
*Proof.* Let  $\mu$  be a metric on  $Y$ . Define a metric  $\varrho$  on  $X$  by  $\varrho(x_1, x_2) = \mu(j(x_1), j(x_2))$ . It is easy to check that the topology on  $X$  is induced by the metric  $\varrho$  (exercise).  $\square$

Let now  $X$  be a space as in Theorem 12.1. In order to show that  $X$  is metrizable it will be enough to construct an embedding  $j: X \rightarrow Y$  where  $Y$  is metrizable. The space  $Y$  will be obtained as a product of topological spaces:

**12.6 Definition.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. The *product topology* on  $\prod_{i \in I} X_i$  is the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \mid U_i \text{ is open in } X_i \text{ and } U_i \neq X_i \text{ for finitely many indices } i \text{ only} \right\}$$

**12.7 Note.** 1) If  $X_1, X_2$  are topological spaces then the product topology on  $X_1 \times X_2$  is the topology induced by the basis  $\mathcal{B} = \{U_1 \times U_2 \mid U_1 \text{ is open in } X_1, U_2 \text{ is open in } X_2\}$ .



2) In general if  $X_1, \dots, X_n$  are topological spaces then the product topology on  $X_1 \times \dots \times X_n$  is the topology generated by the basis  $\mathcal{B} = \{U_1 \times \dots \times U_n \mid U_i \text{ is open in } X_i\}$ .

3) If  $\{X_i\}_{i=1}^\infty$  is an infinitely countable family of topological spaces then the basis of the product topology on  $\prod_{i=1}^\infty X_i$  consists of all sets of the form

$$U_1 \times \dots \times U_n \times X_{n+1} \times X_{n+2} \times X_{n+3} \times \dots$$

where  $n \geq 0$  and  $U_i \subseteq X_i$  is an open set for  $i = 1, \dots, n$ .

**12.8 Proposition.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces and for  $j \in I$  let*

$$p_j: \prod_{i \in I} X_i \rightarrow X_j$$

*be the projection onto the  $j$ -th factor:  $p_j((x_i)_{i \in I}) = x_j$ . Then:*

- 1) for any  $j \in I$  the function  $p_j$  is continuous.
- 2) A function  $f: Y \rightarrow \prod_{i \in I} X_i$  is continuous if and only if the composition  $p_j f: Y \rightarrow X_j$  is continuous for all  $j \in I$

*Proof.* Exercise. □

**12.9 Note.** Notice that the basis  $\mathcal{B}$  given in Definition 12.6 consists of all sets of the form

$$p_{i_1}^{-1}(U_1) \cap \cdots \cap p_{i_n}^{-1}(U_n)$$

where  $i_1, \dots, i_n \in I$  and  $U_{i_1} \subseteq X_{i_1}, \dots, U_{i_n} \subseteq X_{i_n}$  are open sets.

**12.10 Proposition.** If  $X_1, \dots, X_n$  are metrizable spaces then  $X_1 \times \cdots \times X_n$  is also a metrizable space.

*Proof.* Let  $q_i$  be a metric on  $X_i$ . For  $(x_1, \dots, x_n), (x'_1, \dots, x'_n) \in X_1 \times \cdots \times X_n$  define

$$q((x_1, \dots, x_n), (x'_1, \dots, x'_n)) = \sum_{i=1}^n q_i(x_i, x'_i)$$

One can show that  $q$  is a metric, and that the topology induced by this metric is the product topology on  $X_1 \times \cdots \times X_n$  (exercise). □

**12.11 Proposition.** If  $\{X_i\}_{i=1}^\infty$  is an infinite countable family of metrizable spaces then  $\prod_{i=1}^\infty X_i$  is also a metrizable space.

*Proof.* Let  $q_i$  be a metric on  $X_i$ . We can assume that for any  $x, x' \in X_i$  we have  $q_i(x, x') \leq 1$ . Indeed, if  $q_i$  does not have this property then we can replace it by the metric  $q'_i$  given by:

$$q'_i(x, x') = \begin{cases} q_i(x, x') & \text{if } q_i(x, x') \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

The metrics  $q_i$  and  $q'_i$  are equivalent (exercise), and so they define the same topology on the space  $X_i$ .

Given metrics  $q_i$  on  $X_i$  satisfying the above condition define a metric  $q_\infty$  on  $\prod_{i=1}^\infty X_i$  by:

$$q_\infty((x_i), (x'_i)) = \sum_{i=1}^\infty \frac{1}{2^i} q_i(x_i, x'_i)$$

The topology induced by the metric  $q_\infty$  on  $\prod_{i=1}^\infty X_i$  is the product topology (exercise). □

**12.12 Example.** The *Hilbert cube* is the topological space  $[0, 1]^{\aleph_0}$  obtained as the infinite countable product of the closed interval  $[0, 1]$ :

$$[0, 1]^{\aleph_0} = \prod_{i=1}^\infty [0, 1]$$

Elements of  $[0, 1]^{\aleph_0}$  are infinite sequences  $(t_i) = (t_1, t_2, \dots)$  where  $t_i \in [0, 1]$  for  $i = 1, 2, \dots$ . The Hilbert cube is a metric space with a metric  $g$  given by

$$g((t_i), (s_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} |t_i - s_i|$$

Theorem 12.1 will follow from a result on embeddings of topological spaces:

**12.13 Definition.** Let  $X$  be a topological space and let  $\{f_i\}_{i \in I}$  be a family of continuous functions  $f_i: X \rightarrow [0, 1]$ . We say that the family  $\{f_i\}_{i \in I}$  *separates points from closed sets* if for any point  $x_0 \in X$  and any closed set  $A \subseteq X$  such that  $x_0 \notin A$  there is a function  $f_j \in \{f_i\}_{i \in I}$  such that  $f_j(x_0) > 0$  and  $f_j|_A = 0$ .

**12.14 Embedding Lemma.** Let  $X$  be a  $T_1$ -space. If  $\{f_i: X \rightarrow [0, 1]\}_{i \in I}$  is a family that separates points from closed sets then the map

$$f_{\infty}: X \rightarrow \prod_{i \in I} [0, 1]$$

given by  $f_{\infty}(x) = (f_i(x))_{i \in I}$  is an embedding.

**12.15 Note.** If the family  $\{f_i\}_{i \in I}$  in Lemma 12.14 is infinitely countable then  $f_{\infty}$  is an embedding of  $X$  into the Hilbert cube  $[0, 1]^{\aleph_0}$ .

*Proof of Lemma 12.14.* We need to show that the function  $f_{\infty}$  satisfies the following conditions:

- 1)  $f_{\infty}$  is continuous;
- 2)  $f_{\infty}$  is 1-1;
- 3)  $f_{\infty}: X \rightarrow f_{\infty}(X)$  is a homeomorphism.

1) Let  $p_j: \prod_{i \in I} [0, 1] \rightarrow [0, 1]$  be the projection onto the  $j$ -th coordinate. Since  $p_j f_{\infty} = f_j$ , thus  $p_j f_{\infty}$  is a continuous function for all  $j \in I$ . Therefore by Proposition 12.8 the function  $f_{\infty}$  is continuous.

2) Let  $x, y \in X$ ,  $x \neq y$ . Since  $X$  is a  $T_1$ -space the set  $\{y\}$  is closed in  $X$ . Therefore there is a function  $f_j \in \{f_i\}_{i \in I}$  such that  $f_j(x) > 0$  and  $f_j(y) = 0$ . In particular  $f_j(x) \neq f_j(y)$ . Since  $f_j = p_j f_{\infty}$  this gives  $p_j f_{\infty}(x) \neq p_j f_{\infty}(y)$ . Therefore  $f_{\infty}(x) \neq f_{\infty}(y)$ .

3) Let  $U \subseteq X$  be an open set. We need to prove that the set  $f_{\infty}(U)$  is open in  $f(X)$ . It will suffice to show that for any  $x_0 \in U$  there is a set  $V$  open in  $f_{\infty}(X)$  such that  $f_{\infty}(x_0) \in V$  and  $V \subseteq f_{\infty}(U)$ .

Given  $x_0 \in U$  let  $f_j \in \{f_i\}_{i \in I}$  be a function such that  $f_j(x_0) > 0$  and  $f_j|_{X \setminus U} = 0$ . Let  $p_j: \prod_{i \in I} [0, 1] \rightarrow [0, 1]$  be the projection onto the  $j$ -th coordinate. Define

$$V := f_{\infty}(X) \cap p_j^{-1}((0, 1])$$

The set  $V$  is open in  $f_\infty(X)$  since  $p_j^{-1}((0, 1])$  is open in  $\prod_{i \in I} [0, 1]$ . Notice that

$$\begin{aligned} V &= \{f_\infty(x) \mid x \in X \text{ and } p_j f_\infty(x) > 0\} \\ &= \{f_\infty(x) \mid x \in X \text{ and } f_j(x) > 0\} \end{aligned}$$

Since  $f_j(x_0) > 0$  we have  $f_\infty(x_0) \in V$ . Finally, since  $f_j(x) = 0$  for all  $x \in X \setminus U$  we get that  $V \subseteq f_\infty(U)$ .  $\square$

We are now ready to prove the Urysohn Metrization Theorem:

*Proof of Theorem 12.1.* Let  $\mathcal{B} = \{V_i\}_{i=1}^\infty$  be a countable basis of  $X$ , and let  $S$  the set given by

$$S := \{(i, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \bar{V}_i \subseteq V_j\}$$

If  $(i, j) \in S$  then the sets  $\bar{V}_i$  and  $X \setminus V_j$  are closed and disjoint, so by the Urysohn Lemma 10.1 there is a continuous function  $f_{ij}: X \rightarrow [0, 1]$  such that

$$f_{ij}(x) = \begin{cases} 1 & \text{if } x \in \bar{V}_i \\ 0 & \text{if } x \in X \setminus V_j \end{cases}$$

We will show that the family  $\{f_{ij}\}_{(i,j) \in S}$  separates points from closed sets. Take  $x_0 \in X$  and let  $A \subseteq X$  be an closed set such that  $x_0 \notin A$ . Since  $\mathcal{B} = \{V_i\}_{i=1}^\infty$  is a basis of  $X$  there is  $V_j \in \mathcal{B}$  such that  $x_0 \in V_j$  and  $V_j \subseteq X \setminus A$ . Using Lemma 10.3 we also obtain that there exists  $V_i \in \mathcal{B}$  such that  $x_0 \in V_i$  and  $\bar{V}_i \subseteq V_j$ . We have  $f_{ij}(x_0) = 1$ . Also, since  $A \subseteq X \setminus V_j$  we have  $f_{ij}|_A = 0$ .

By the Embedding Lemma 12.14 the family  $\{f_{ij}\}_{(i,j) \in S}$  defines an embedding

$$f_\infty: X \rightarrow \prod_{(i,j) \in S} [0, 1]$$

The set  $S$  is countable. If it is infinite then  $\prod_{(i,j) \in S} [0, 1] \cong [0, 1]^{\aleph_0}$  which is a metrizable space by Proposition 12.11. If  $S$  is finite then  $\prod_{(i,j) \in S} [0, 1] \cong [0, 1]^N$  for some  $N \geq 0$  and  $[0, 1]^N$  which is again metrizable by Proposition 12.10.

The statement of Theorem 12.1 follows now from Lemma 12.5.  $\square$

One can show that the following holds:

**12.16 Proposition.** *Every second countable regular space is normal.*

*Proof.* Exercise.  $\square$

As a consequence Theorem 12.1 can be reformulated as follows:

**12.17 Urysohn Metrization Theorem (v.2).** *Every second countable regular space is metrizable.*

While every metrizable space is normal (and regular) such spaces do not need to be second countable. For example, any discrete space  $X$  is metrizable, but if  $X$  consists of uncountably many points it does not have a countable basis (Exercise 4.10). This means that the converse of the Urysohn Metrization Theorem does not hold. However, this theorem can be generalized to give conditions that are both sufficient and necessary for metrizability of a space. We finish this chapter by giving the statement of such result without proof.

**12.18 Definition.** Let  $X$  be a topological space. A collection  $\mathcal{U} = \{U_i\}_{i \in I}$  of open sets in  $X$  is *locally finite* if each point  $x \in X$  has an open neighborhood  $V_x$  such that  $V_x \cap U_i \neq \emptyset$  for finitely many  $i \in I$  only.

A collection  $\mathcal{U}$  is *countably locally finite* if it can be decomposed into a countable union  $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$  where each collection  $\mathcal{U}_n$  is locally finite.

**12.19 Nagata-Smirnov Metrization Theorem.** *Let  $X$  be a topological space. The following conditions are equivalent:*

- 1)  $X$  is metrizable.
- 2)  $X$  is regular and it has a basis which is countably locally finite.

## Exercises to Chapter 12

**E12.1 Exercise.** Show that the product topology on  $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$  is the same as the topology induced by the Euclidean metric.

**E12.2 Exercise.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. The *box topology* on  $\prod_{i \in I} X_i$  is the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \mid U_i \text{ is open in } X_i \right\}$$

Notice that for products of finitely many spaces the box topology is the same as the product topology, but that it differs if we take infinite products.

Let  $X = \prod_{n=1}^{\infty} [0, 1]$  be the product of countably many copies of the interval  $[0, 1]$ . Consider  $X$  as a topological space with the box topology. Show that the map  $f: [0, 1] \rightarrow X$  given by  $f(t) = (t, t, t, \dots)$  is not continuous.

**E12.3 Exercise.** Prove Proposition 12.8

**E12.4 Exercise.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces and for  $i \in I$  let  $A_i$  be a closed set in

$X_i$ . Show that the set  $\prod_{i \in I} A_i$  is closed in the product topology on  $\prod_{i \in I} X_i$ .

**E12.5 Exercise.** Let  $X$  and  $Y$  be non-empty topological spaces. Show that the space  $X \times Y$  is connected if and only if  $X$  and  $Y$  are connected.

**E12.6 Exercise.** Assume that  $X, Y$  are spaces such that  $\mathbb{R} \cong X \times Y$ . Show that either  $X$  or  $Y$  consists of only one point.

**E12.7 Exercise.** Let  $X, Y$  be topological spaces. For a (not necessarily continuous) function  $f: X \rightarrow Y$  the *graph* of  $f$  is the subspace  $\Gamma(f)$  of  $X \times Y$  given by

$$\Gamma(f) = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

Show that if  $Y$  is a Hausdorff space and  $f: X \rightarrow Y$  is a continuous function then  $\Gamma(f)$  is closed in  $X \times Y$ .

**E12.8 Exercise.** Let  $X_1, X_2$  be topological spaces, and for  $i = 1, 2$  let  $p_i: X_1 \times X_2 \rightarrow X_i$  be the projection map.

a) Show that if a set  $U \subseteq X_1 \times X_2$  is open in  $X_1 \times X_2$  then  $p_i(U)$  is open in  $X_i$ .

b) Is it true that if  $A \subseteq X_1 \times X_2$  is a closed set then  $p_i(A)$  must be closed in  $X_i$ ? Justify your answer.

**E12.9 Exercise.** The goal of this exercise is to complete the proof of Proposition 12.10. For  $i = 1, 2, \dots, n$  let  $(X_i, \varrho_i)$  be a metric space. Let  $\varrho$  be a metric the Cartesian product  $X_1 \times \dots \times X_n$  given by

$$\varrho((x_1, \dots, x_n), (x'_1, \dots, x'_n)) = \sum_{i=1}^n \varrho_i(x_i, x'_i)$$

Show that the topology defined by  $\varrho$  is the same as the product topology.

**E12.10 Exercise.** The goal of this exercise is to complete the proof of Proposition 12.11. For  $i = 1, 2, \dots$  let  $(X_i, \varrho_i)$  be a metric space such that  $\varrho_i(x, x') \leq 1$  for all  $x, x' \in X_i$ . Let  $\varrho_\infty$  be a metric the Cartesian product  $\prod_{i=1}^\infty X_i$  given by

$$\varrho_\infty((x_i), (x'_i)) = \sum_{i=1}^\infty \frac{1}{2^i} \varrho_i(x_i, x'_i)$$

Show that the topology defined by  $\varrho_\infty$  is the same as the product topology.

**E12.11 Exercise.** The goal of this exercise is to give a proof of Proposition 12.16. Let  $X$  be a second countable regular space and let  $A, B \subseteq X$  be closed sets such that  $A \cap B = \emptyset$ .

a) Show that there exist countable families of open sets  $\{U_1, U_2, \dots\}$  and  $\{V_1, V_2, \dots\}$  such that

(i)  $A \subseteq \bigcup_{i=1}^\infty U_i$  and  $B \subseteq \bigcup_{i=1}^\infty V_i$

(ii) for all  $i \geq 1$  we have  $\overline{U}_i \cap B = \emptyset$  and  $\overline{V}_i \cap A = \emptyset$

b) For  $n \geq 1$  define

$$U'_n := U_n \setminus \bigcup_{i=1}^n \overline{V}_i \quad \text{and} \quad V'_n := V_n \setminus \bigcup_{i=1}^n \overline{U}_i$$

Let  $U' = \bigcup_{n=1}^{\infty} U'_n$  and  $V' = \bigcup_{n=1}^{\infty} V'_n$ . Show that  $U'$  and  $V'$  are open sets, that  $A \subseteq U'$  and  $B \subseteq V'$ , and that  $U' \cap V' = \emptyset$ .

**E12.12 Exercise.** Let  $\mathbb{R}_{disc}$  denote the real line with the discrete topology and let  $X = \prod_{n=1}^{\infty} \mathbb{R}_{disc}$ .

a) Show that  $X$  is not second countable.

b) By Proposition 12.11 we know that  $X$  is a metrizable space. Verify this fact without using Proposition 12.11, but using instead only topological properties of  $X$  and the Nagata-Smirnov Metrization Theorem 12.19.