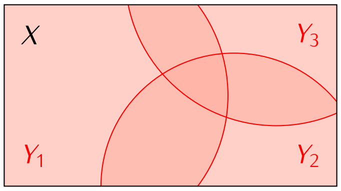
14 | Compact Spaces

14.1 Definition. Let X be a topological space. A *cover* of X is a collection $\mathcal{Y} = \{Y_i\}_{i\in I}$ of subsets of X such that $\bigcup_{i\in I} Y_i = X$.



If the sets Y_i are open in X for all $i \in I$ then $\mathcal{Y}$ is an *open cover* of X. If $\mathcal{Y}$ consists of finitely many sets then $\mathcal{Y}$ is a *finite cover* of X.

14.2 Definition. Let $\mathcal{Y} = \{Y_i\}_{i\in I}$ be a cover of X. A *subcover* of $\mathcal{Y}$ is cover $\mathcal{Y}'$ of X such that every element of $\mathcal{Y}'$ is in $\mathcal{Y}$.

14.3 Example. Let $X = \mathbb{R}$. The collection

$$\mathcal{Y} = \{(m, n) \subseteq \mathbb{R} \mid m, n \in \mathbb{Z},\ m < n\}$$

is an open cover of $\mathbb{R}$, and the collection

$$\mathcal{Y}' = \{(-n, n) \subseteq \mathbb{R} \mid n = 1, 2, \dots\}$$

is a subcover of $\mathcal{Y}$.

14.4 Definition. A space X is *compact* if every open cover of X contains a finite subcover.

14.5 Example. A discrete topological space X is compact if and only if X consists of finitely many points.

14.6 Example. Let X be a subspace of $\mathbb{R}$ given by

$$X = \{0\} \cup \{ \tfrac{1}{n} \mid n = 1, 2, \dots \}$$

The space X is compact. Indeed, let $\mathcal{U} = \{U_i\}_{i \in I}$ be any open cover of X and let $0 \in U_0$. Then there exists $N > 0$ such that $\frac{1}{n} \in U_{i_0}$ for all $n > N$. For $n = 1, \dots, N$ let $U_{i_n} \in \mathcal{U}$ be a set such that $\frac{1}{n} \in U_{i_n}$. We have:

$$X = U_{i_0} \cup U_{i_1} \cup \dots \cup U_{i_N}$$

so $\{U_{i_0}, U_{i_1}, \dots, U_{i_N}\}$ is a finite subcover of $\mathcal{U}$.

14.7 Example. The real line $\mathbb{R}$ is not compact since the open cover

$$\mathcal{Y} = \{(n-1, n+1) \subseteq \mathbb{R} \mid n \in \mathbb{Z}\}$$

does not have any finite subcover.

14.8 Proposition. *For any $a < b$ the closed interval $[a, b] \subseteq \mathbb{R}$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $[a, b]$ and let

$$A = \{x \in [a, b] \mid \text{the interval } [a, x] \text{ can be covered by a finite number of elements of } \mathcal{U}\}$$

Let $x_0 := \sup A$.

Step 1. We will show that $x_0 > a$. Indeed, let $U \in \mathcal{U}$ be a set such that $a \in U$. Since U is open we have $[a, a + \varepsilon) \subseteq U$ for some $\varepsilon > 0$. It follows that $x \in A$ for all $x \in [a, a + \varepsilon)$. Therefore $x_0 \geq a + \varepsilon$.

Step 2. Next, we will show that $x_0 \in A$. Let $U_0 \in \mathcal{U}$ be a set such that $x_0 \in U_0$. Since U_0 is open and $x_0 > a$ there exists $\varepsilon_1 > 0$ such that $(x_0 - \varepsilon_1, x_0] \subseteq U_0$. Also, since $x_0 = \sup A$ there is $x \in A$ such that $x \in (x_0 - \varepsilon_1, x_0]$. Notice that

$$[a, x_0] = [a, x] \cup (x_0 - \varepsilon_1, x_0]$$

By assumption the interval $[a, x]$ can be covered by a finite number of sets from $\mathcal{U}$ and $(x_0 - \varepsilon_1, x_0]$ is covered by $U_0 \in \mathcal{U}$. As a consequence $[a, x_0]$ can be covered by a finite number of elements of $\mathcal{U}$, and so $x_0 \in A$.

Step 3. In view of Step 2 it suffices to show that $x_0 = b$. To see this take again $U_0 \in \mathcal{U}$ to be a set such that $x_0 \in \mathcal{U}$. If $x_0 < b$ then there exists $\varepsilon_2 > 0$ such that $[x_0, x_0 + \varepsilon_2) \subseteq U_0$. Notice that for any $x \in (x_0, x_0 + \varepsilon_2)$ the interval $[a, x]$ can be covered by a finite number of elements of $\mathcal{U}$, and thus $x \in A$. Since $x > x_0$ this contradicts the assumption that $x_0 = \sup A$.

$\square$

14.9 Proposition. *Let $f \colon X \to Y$ be a continuous function. If X is compact and f is onto then Y is compact.*

Proof. Exercise. □

14.10 Corollary. *Let $f\colon X \to Y$ be a continuous function. If $A \subseteq X$ is compact then $f(A) \subseteq Y$ is compact.*

Proof. The function $f|_A\colon A \to f(A)$ is onto, so this follows from Proposition 14.9. □

14.11 Corollary. *Let X, Y be topological spaces. If X is compact and $Y \cong X$ then Y is compact.*

Proof. Follows from Proposition 14.9. □

14.12 Example. For any $a < b$ the open interval $(a, b) \subseteq \mathbb{R}$ is not compact since $(a, b) \cong \mathbb{R}$.

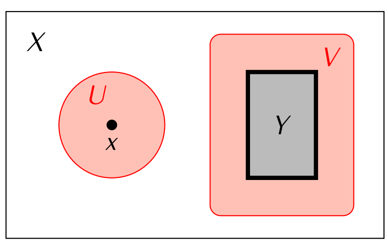
14.13 Proposition. *Let X be a compact space. If Y is a closed subspace of X then Y is compact.*

Proof. Exercise. □

14.14 Proposition. *Let X be a Hausdorff space and let $Y \subseteq X$. If Y is compact then it is closed in X.*

Proposition 14.14 is a direct consequence of the following fact:

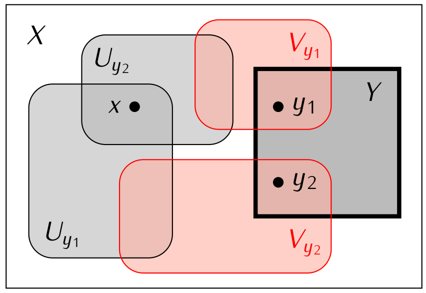
14.15 Lemma. *Let X be a Hausdorff space, let $Y \subseteq X$ be a compact subspace, and let $x \in X \smallsetminus Y$. There exists open sets $U, V \subseteq X$ such that $x \in U$, $Y \subseteq V$ and $U \cap V = \varnothing$.*



Proof. Since X is a Hausdorff space for any point $y \in Y$ there exist open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \varnothing$. Notice that $Y \subseteq \bigcup_{y\in Y} V_y$. Since Y is compact we can find a finite number of points $y_1, \dots, y_n \in Y$ such that

$$Y \subseteq V_{y_1} \cup \dots \cup V_{y_n}$$

Take $V = V_{y_1} \cup \dots \cup V_{y_n}$ and $U := U_{y_1} \cap \dots \cap U_{y_n}$.



□

Proof of Proposition 14.14. By Lemma 14.15 for each point $x \in X \smallsetminus Y$ we can find an open set $U_x \subseteq X$ such that $x \in U_x$ and $U_x \subseteq X \smallsetminus Y$. Therefore $X \smallsetminus Y$ is open and so Y is closed. □

14.16 Corollary. *Let X be a compact Hausdorff space. A subspace $Y \subseteq X$ is compact if and only if Y is closed in X.*

Proof. Follows from Proposition 14.13 and Proposition 14.14. □

14.17 Proposition. *Let $f: X \to Y$ be a continuous function, where X is a compact space and Y is a Hausdorff space. For any closed set $A \subseteq X$ the set $f(A)$ is closed in Y.*

Proof. Let $A \subseteq X$ be a closed set. By Proposition 14.13 A is a compact space and thus by Corollary 14.10 $f(A)$ is a compact subspace of Y. Since Y is a Hausdorff space, using Proposition 14.14 we obtain that $f(A)$ is closed in Y. □

14.18 Proposition. *Let $f: X \to Y$ be a continuous bijection. If X is a compact space and Y is a Hausdorff space then f is a homeomorphism.*

Proof. This follows from Proposition 6.12 and Proposition 14.18. □

14.19 Theorem. *If X is a compact Hausdorff space then X is normal.*

Proof. Step 1. We will show first that X is a regular space (9.9). Let $A \subseteq X$ be a closed set and let $x \in X \smallsetminus A$. We need to show that there exists open sets $U, V \subseteq X$ such that $x \in U$, $A \subseteq V$ and $U \cap V = \varnothing$. Notice that by Proposition 14.13 the set A is compact. Since X is Hausdorff existence of the sets U and V follows from Lemma 14.15.

Step 2. Next, we show that X is normal. Let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. By Step 1 for every $x \in A$ we can find open sets U_x and V_x such that $x \in U_x$, $B \subseteq V_x$ and $U_x \cap V_x = \varnothing$. The collection $\mathcal{U} = \{U_x\}_{x \in A}$ is an open cover of A. Since A is compact there is a finite number of points

$x_1, \dots, x_m \in A$ such that $\{U_{x_1}, \dots, U_{x_m}\}$ is a cover of A. Take $U := \bigcup_{i=1}^m U_{x_i}$ and $V := \bigcap_{i=1}^m V_{x_i}$. Then U and V are open sets, $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$. □

Exercises to Chapter 14

E14.1 Exercise. Prove Proposition 14.9.

E14.2 Exercise. Prove Proposition 14.13.

E14.3 Exercise. Let X be a Hausdorff space and let $A \subseteq X$. Show that the following conditions are equivalent:

(i)) A is compact

(ii)) A is closed in X and in any open cover $\{U_i\}_{i\in I}$ of X there exists a finite number of sets $U_{i_1}, \dots, U_{i_n}$ such that $A \subseteq \bigcup_{k=1}^n U_{i_k}$.

E14.4 Exercise. a) Let X be a compact space and for $i = 1, 2, \dots$ let $A_i \subseteq X$ be a non-empty closed set. Show that if $A_{i+1} \subseteq A_i$ for all i then $\bigcap_{i=1}^\infty A_i \neq \varnothing$.

b) Give an example of a (non-compact) space X and closed non-empty sets $A_i \subseteq X$ satisfying $A_{i+1} \subseteq A_i$ for $i = 1, 2, \dots$ such that $\bigcap_{i=1}^\infty A_i = \varnothing$.

E14.5 Exercise. a) Let X be a compact Hausdorff space and for $i = 1, 2, \dots$ let $A_i \subseteq X$ be a closed, connected set. Show that if $A_{i+1} \subseteq A_i$ for all i then $\bigcap_{i=1}^\infty A_i$ is connected.

b) Give an example of a space X and subspaces $A_1 \supseteq A_2 \supseteq \dots$ such that A_i is connected and closed in X for each i, but $\bigcap_{i=1}^\infty A_i$ is not connected.

E14.6 Exercise. The goal of this exercise is to show that if $f\colon X \to \mathbb{R}$ is a continuous function and X is a compact space then there exist points $x_1, x_2 \in X$ such that $f(x_1)$ is the minimum value of f and $f(x_2)$ is the maximum value.

Let X be a compact space and let $f\colon X \to \mathbb{R}$ be a continuous function.

a) Show that there exists $C > 0$ such that $|f(x)| < C$ for all $x \in X$.

b) By part a) there exists $C > 0$ such that $f(X) \subseteq [-C, C]$. This implies that $\inf f(X) \neq -\infty$ and $\sup f(X) \neq +\infty$. Show that there are points $x_1, x_2 \in X$ such that $f(x_1) = \inf f(X)$ and that $f(x_2) = \sup f(X)$.

E14.7 Exercise. Let (X, ϱ) be a compact metric space, and let $f\colon X \to X$ be a function such that $\varrho(f(x), f(y)) < \varrho(x, y)$ for all $x, y \in X$, $x \neq y$.

a) Show that the function $\varphi\colon X \to \mathbb{R}$ given by $\varphi(x) = \varrho(x, f(x))$ is continuous.

b) Show that there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

E14.8 Exercise. Let $f\colon X \to Y$ be a continuous map such for any closed set $A \subseteq X$ the set $f(A)$ is closed in Y.

a) Let $y \in Y$. Show that if $U \subseteq X$ is an open set and $f^{-1}(y) \subseteq U$ then there exists an open set $V \subseteq Y$ such that $y \in V$ and $f^{-1}(V) \subseteq U$.

b) Show that if Y is compact and $f^{-1}(y)$ is compact for each $y \in Y$ then X is compact.

E14.9 Exercise. Let X, Y be topological spaces, and let $p_1\colon X \times Y \to X$ be the projection map: $p_1(x, y) = x$. Show that if Y is compact then for any closed set $A \subseteq X \times Y$ the set $p_1(A) \subseteq X$ is closed in X.

E14.10 Exercise. A continuous function $f\colon X \to Y$ is a *local homeomorphism* if for each point $x \in X$ there exists an open neighborhood $U_x \subseteq X$ such that $f(U_x)$ is open in Y and $f|_{U_x}\colon U_x \to f(U_x)$ is a homeomorphism.

a) Assume that $f\colon X \to Y$ is a local homeomorphism where X is a compact space. Show that for each $y \in Y$ the set $f^{-1}(y)$ consists of finitely many points.

b) Assume that $f\colon X \to Y$ is a local homeomorphism where X is a compact Hausdorff space and Y is a Hausdorff space. Let $y \in Y$ be a point such that $f^{-1}(y)$ consists of n points. Show that there exists an open set $V \subseteq Y$ such that $y \in V$ and that for each $y' \in V$ the set $f^{-1}(y')$ consists of n points.